Generalization of Adamchik's Formulas

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This note discusses generalizations of formulas in "A Class of Logarithmic Integrals" by Victor Adamchik. Adamchik's paper proves 6 different general propositions allowing Adamchik to solve integrals such as

$$\int_0^1 \frac{x}{1+x^4} \log\left(\log\left(\frac{1}{x}\right)\right) dx = \frac{\pi}{4} \log\left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)$$

$$\int_0^1 \frac{\sqrt{x}}{(1+x)^2} \log\left(\log\left(\frac{1}{x}\right)\right) dx$$
$$= \frac{\gamma}{2} - \frac{3}{2}\log(2) + \log\left(\frac{4\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right) + \frac{\pi}{2} \log\left(\frac{2\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)$$

$$\int_0^1 \frac{\sqrt{x}}{(x+1)^3} \log\left(\log\left(\frac{1}{x}\right)\right) dx = -\frac{G}{2\,\pi} + \frac{\pi}{8} \log\left(\frac{2\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right)$$

Generally, I like Adamchik's paper. The most interesting proposition is Proposition 1 which Adamchik attributes to G. Almkvist and A. Meurman. The proofs for Adamchik's Proposition 3, Proposition 5, and Proposition 6 are more drawn out and tortured than necessary. My theorem below is proved in less space and generalizes half the propositions in Adamchik's paper.

1. Definition of Two Functions

Definition. For convenience, define

$$\overline{\zeta}(a,b) = 2^a\,\Phi\left(-1,a,2\,b\right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\left(\frac{k}{2}+b\right)^a} = \zeta(a,b) - \zeta\left(a, b+\frac{1}{2}\right)$$

$$\overline{\psi}(a) = \psi(a) - \psi\left(a+\frac{1}{2}\right)$$

Function $\overline{\zeta}$ has properties

$$\overline{\zeta}(1,x) = -\overline{\psi}(x) \qquad \overline{\zeta}\left(-n,x\right) = \frac{\mathrm{B}_{\mathrm{n+1}}\left(x+\frac{1}{2}\right) - \mathrm{B}_{\mathrm{n+1}}(x)}{n+1}$$

$$\overline{\zeta}'\left(0,x\right) = \log\left(\frac{\Gamma(x)}{\Gamma\left(x+\frac{1}{2}\right)}\right)$$

2. Generalization of Adamchik's Propositions 3, 5, and 6

The following theorem generalizes Adamchik's Proposition 3, Proposition 5, and Proposition 6.

Theorem.

$$\int_0^1 \frac{x^{p-1}}{\left(1+x^n\right)^q} \log\left(\log\left(\frac{1}{x}\right)\right)\,dx$$

$$= -\left(\gamma + \log\left(2\,n\right)\right) \sum_{j=0}^{q-1} c_{qj}\,\overline{\zeta}\left(1-j, \frac{p}{2\,n}\right)$$

$$+ \sum_{j=0}^{q-1} c_{qj}\,\overline{\zeta}'\left(1-j, \frac{p}{2\,n}\right)$$

where coefficients c_{qj} are determined by

$$\mathrm{c_q}(x) = \frac{\left(2\,x - \frac{p}{n} + 1\right)_{q-1}}{(q-1)!\,(2\,n)} = \sum_{j=0}^{q-1} c_{qj}\,x^j$$

Proof. Since

$$\frac{(k+1)_{q-1}}{(q-1)!\,(2\,n)} = \mathrm{c_q}\left(\frac{k}{2} + \frac{p}{2\,n}\right)$$

$$\frac{1}{(1+x)^q} = \sum_{k=0}^{\infty} \binom{-q}{k} x^{-q-k} = \sum_{k=0}^{\infty} \frac{(-1)^k\,(k+1)_{q-1}}{(q-1)!}\,x^{-q-k}$$

We get

$$\int_0^\infty \frac{y^\epsilon\,e^{(q\,n-p)\,y}}{(1+e^{n\,y})^q}\,dy$$
$$= \sum_{k=0}^{\infty}\left(\frac{(-1)^k\,(k+1)_{q-1}}{(q-1)!}\int_0^\infty y^\epsilon\,e^{-(k\,n+p)\,y}\,dy\right)$$
$$= \frac{\Gamma(1+\epsilon)}{(q-1)!}\sum_{k=0}^{\infty}\frac{(-1)^k\,(k+1)_{q-1}}{(k\,n+p)^{1+\epsilon}}$$
$$= \frac{\Gamma(1+\epsilon)}{(2\,n)^\epsilon}\sum_{k=0}^{\infty}\frac{(-1)^k\,(k+1)_{q-1}}{(q-1)!\,(2\,n)\,\left(\frac{k}{2}+\frac{p}{2\,n}\right)^{1+\epsilon}}$$
$$= \frac{\Gamma(1+\epsilon)}{(2\,n)^\epsilon}\sum_{j=0}^{q-1} c_{qj}\,\overline{\zeta}\left(1-j+\epsilon,\frac{p}{2\,n}\right)$$

Therefore,

$$\int_0^1 \frac{x^{p-1}}{(1+x^n)^q} \log\left(\log\left(\frac{1}{x}\right)\right) dx = \int_0^\infty \frac{e^{(q\,n-p)\,y}}{(1+e^{n\,y})^q} \log(y)\,dy$$

$$= \left(\frac{\partial}{\partial\epsilon}\left(\int_0^\infty \frac{y^\epsilon\, e^{(q\,n-p)\,y}}{(1+e^{n\,y})^q}\,dy\right)\right)\Big|_{\epsilon=0}$$

$$= \left(\frac{\partial}{\partial\epsilon}\left(\frac{\Gamma(1+\epsilon)}{(2\,n)^\epsilon}\sum_{j=0}^{q-1} c_{qj}\,\overline{\zeta}\left(1-j+\epsilon, \frac{p}{2\,n}\right)\right)\right)\Big|_{\epsilon=0}$$

$$= -\left(\gamma + \log(2\,n)\right)\sum_{j=0}^{q-1} c_{qj}\,\overline{\zeta}\left(1-j, \frac{p}{2\,n}\right)$$

$$+ \sum_{j=0}^{q-1} c_{qj}\,\overline{\zeta}'\left(1-j, \frac{p}{2\,n}\right)$$

Comment. The first few $c_q(x)$ are

$$\mathrm{c}_1(x) = \frac{1}{2\,n}$$

$$\mathrm{c}_2(x) = \frac{1}{n}\,x + \frac{n-p}{2\,n^2}$$

$$\mathrm{c}_3(x) = \frac{1}{n}\,x^2 + \frac{3\,n-2\,p}{2\,n^2}\,x + \frac{(2\,n-p)\,(n-p)}{4\,n^3}$$

$$\mathrm{c}_4(x) = \frac{2}{3\,n}\,x^3 + \frac{2\,n-p}{n^2}\,x^2 + \frac{11\,n^2-12\,n\,p+3\,p^2}{6\,n^3}\,x + \frac{(n-p)\,(2\,n-p)\,(-p+3\,n)}{12\,n^4}$$

3. Adamchik's Proposition 4

The proof of Adamchik's Proposition 4 can also be simpler:

Theorem.

$$
\begin{aligned}
&\int_0^1 x^{p-1}\,\frac{1-x}{1-x^n}\log\left(\log\left(\frac{1}{x}\right)\right)\,dx\\
&\quad=\frac{\gamma+\log(n)}{n}\,\left(\psi\left(\frac{p}{n}\right)-\psi\left(\frac{p+1}{n}\right)\right)\\
&\qquad+\frac{1}{n}\,\left(\overline{\zeta}'\left(1,\frac{p}{n}\right)-\overline{\zeta}'\left(1,\frac{p+1}{n}\right)\right)
\end{aligned}
$$

Proof.

$$
\begin{aligned}
&\int_0^\infty y^{\epsilon}\,e^{-p\,y}\,\frac{1-e^{-y}}{1-e^{-n\,y}}\,dy\\
&\quad=\int_0^\infty y^{\epsilon}\,\left(e^{-p\,y}-e^{-(p+1)\,y}\right)\sum_{k=0}^{\infty}e^{-n\,k\,y}\,dy\\
&\quad=\sum_{k=0}^{\infty}\left(\int_0^\infty y^{\epsilon}\,e^{-(k\,n+p)\,y}\,dy-\int_0^\infty y^{\epsilon}\,e^{-(k\,n+p+1)\,y}\,dy\right)\\
&\quad=\sum_{k=0}^{\infty}\left(\frac{\Gamma\left(1+\epsilon\right)}{\left(k\,n+p\right)^{1+\epsilon}}-\frac{\Gamma\left(1+\epsilon\right)}{\left(k\,n+p+1\right)^{1+\epsilon}}\right)\\
&\quad=\frac{\Gamma\left(1+\epsilon\right)}{n^{1+\epsilon}}\,\left(\zeta\left(1+\epsilon,\frac{p}{n}\right)-\zeta\left(1+\epsilon,\frac{p+1}{n}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\int_0^1 x^{p-1} \frac{1-x}{1-x^n} \log\left(\log\left(\frac{1}{x}\right)\right) dx &= \int_0^\infty e^{-p\,y} \frac{1-e^{-y}}{1-e^{-n\,y}} \log(y)\, dy \\
&= \left(\frac{\partial}{\partial\epsilon}\left(\int_0^\infty y^\epsilon\, e^{-p\,y} \frac{1-e^{-y}}{1-e^{-n\,y}}\, dy\right)\right)\Bigg|_{\epsilon=0} \\
&= \left(\frac{\partial}{\partial\epsilon}\left(\frac{\Gamma\left(1+\epsilon\right)}{n^{1+\epsilon}}\left(\zeta\left(1+\epsilon, \frac{p}{n}\right) - \zeta\left(1+\epsilon, \frac{p+1}{n}\right)\right)\right)\right)\Bigg|_{\epsilon=0} \\
&= \frac{\gamma+\log(n)}{n}\left(\psi\left(\frac{p}{n}\right) - \psi\left(\frac{p+1}{n}\right)\right) \\
&\quad + \frac{1}{n}\left(\overline{\zeta}'\left(1, \frac{p}{n}\right) - \overline{\zeta}'\left(1, \frac{p+1}{n}\right)\right)
\end{aligned}
$$

4. References

The first reference is Victor Adamchik's paper. The other papers also discuss expression of definite integrals in terms of derivatives of products of Γ and ζ functions.

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